

The structure of singularities in inhomogeneous cosmological models.

Alan D. Rendall, Max-Planck-Institut für Gravitationsphysik,
Schlaatzweg 1, 14473 Potsdam, Germany

February 7, 2008

Abstract

Recent progress in understanding the structure of cosmological singularities is reviewed. The well-known picture due to Belinskii, Khalatnikov and Lifschitz (BKL) is summarized briefly and it is discussed what existing analytical and numerical results have to tell us about the validity of this picture. If the BKL description is correct then most cosmological singularities are complicated. However there are some cases where it predicts simple singularities. These cases should be particularly amenable to mathematical investigation and the results in this direction which have been achieved so far are described.

1 Introduction

The purpose of this article is to survey some recent progress in understanding the structure of cosmological singularities. Many years ago, Belinskii, Khalatnikov and Lifshitz (BKL) proposed a picture of singularities in general inhomogeneous spacetimes[2]. This was based on formal calculations which until recently had never been supported by either rigorous mathematical proofs or careful numerical calculations. This picture will be referred to in the following as the BKL scenario.

A key element of the BKL scenario is that near a singularity the evolution at different spatial points decouples. This of course requires a specification of what is meant by a spatial point. As a first attempt at doing this we may suppose that a suitable local coordinate system is used in the definition. The decoupled evolution then corresponds to a spatially homogeneous model at each spacetime point. A second element is that generically the matter content of spacetime should have a negligible effect on the dynamics near the singularity. Thus the essentials of the dynamics in the generic case should be described by vacuum spacetimes. This is why vacuum models are believed to be relevant to the study of cosmological singularities even although the energy density blows up

there. (This can still have a negligible dynamical effect if other quantities in the Einstein equations blow up even faster.) A third element is a description of the dynamics of the most general spatially homogeneous models near the singularity, which is supposed to be modelled by a spacetime of Bianchi type IX. Combining the vacuum condition with the Bianchi type IX condition leads to the Mixmaster solution, which thus takes on a central role in these considerations. A characteristic feature of the Mixmaster solution is that it exhibits complicated oscillatory behaviour.

It should be noted that the statement that the matter is generically negligible near the singularity is subject to certain restrictions. For instance, within the BKL scenario a massless scalar field as matter model remains important up to the singularity and kills Mixmaster oscillations. As we will see later, an electromagnetic field can remain important near the singularity, causing Mixmaster-like behaviour to appear in symmetry classes which show no Mixmaster behaviour in vacuum. There are well-known criteria for deciding which matter models can be neglected and what will be their effect if they cannot but these will not be discussed here. The role of Bianchi type IX is that it and Bianchi type VIII, which has a very similar behaviour near the singularity, are supposed to be the most general spatially homogeneous models. It seems, however, that there is a class of Bianchi type VI_h models with $h = -1/9$ which are equally general and their role in the BKL scenario remains to be clarified. (See [11], section 8.1, for a discussion of this.)

Up to now the BKL scenario is neither supported nor invalidated by rigorous mathematical arguments. In fact even the dynamics of the Mixmaster solution, which serves as a basic model in the scheme, is not understood mathematically. Some partial results, together with a description of some of the open questions can be found in [8]. On the other hand, heuristic and numerical work combine to give good evidence for the BKL scenario in the spatially homogeneous case.

In the spatially inhomogeneous case numerical approaches had, until recently, been no more successful than mathematical ones. The decoupling and pointwise Mixmaster behaviour predicted by BKL had not been reliably seen numerically in any spatially inhomogeneous case. There is an important reason for this which will now be explained. One approach to looking for a simple example would be to start with Mixmaster initial data and do a spatially inhomogeneous perturbation while keeping as much symmetry as possible, so as to make the problem as simple as possible. Unfortunately, the Mixmaster model and its close relative of Bianchi type VIII have the property that no two linearly independent Killing vectors commute. Thus it is impossible to perturb away from three Killing vectors without going all the way down to one Killing vector. The intermediate case of two Killing vectors is not attainable. In this way one is led to a numerical problem where the effective space dimension is two. It is very difficult to produce numerical results which can be trusted in a context which combines Mixmaster-like behaviour and two spatial dimensions.

A possible way round this difficulty was suggested in [9]. This arises from

the fact that solutions of Bianchi type VI_0 with electromagnetic fields show Mixmaster behaviour (see [7]) and that, since the Lie algebra which defines the symmetry of Bianchi type VI_0 has a two-dimensional abelian subalgebra, solutions of this type can be perturbed to solutions with two Killing vectors. (The discussion in [9] suffers from the misconception that matter is required in these models. Actually the source-free Einstein-Maxwell equations suffice.) In this way inhomogeneous models are obtained where the effective space dimension is one.

This possibility has been exploited recently by Weaver, Isenberg and Berger [12]. They showed numerically that, in a class of solutions of the Einstein-Maxwell equations, independent Mixmaster-like oscillations are observed at different spatial points as the singularity is approached, agreeing with the BKL picture in that case. Thus for the first time a solid confirmation of the applicability of the BKL scenario to a class of inhomogeneous spacetimes showing Mixmaster behaviour has been obtained.

2 Simple Singularities

There are special situations in which the BKL picture predicts relatively simple singularities and in this case one may hope to prove rigorous theorems on their structure. Recently results of this kind have been proved using a class of singular partial differential equations, the Fuchsian equations. An introduction to the theory of these equations can be found in [5]. The basic idea in applying these techniques to construct singular solutions of partial differential equations is as follows. Write the solution u as $u_0 + u_1$, where u_0 is an explicit expression which is singular and u_1 is unknown but supposed to be regular. Now rewrite the original equation in terms of the unknown u_1 . The resulting equation for u_1 will in general be singular, even if the original equation was regular. In other words the task of finding singular solutions of a regular equation is transformed into that of finding regular solutions of a singular equation. In favourable cases the singular equation is of a special (Fuchsian) form and it is possible to prove the existence of a unique solution for prescribed singular part u_0 . The end result is that one can solve a sort of Cauchy problem with data on the singularity.

Fuchsian techniques can be used to give statements about the singularities of general solutions in a particular class on different levels. In order of increasing sophistication these are as follows. The first level is to prove existence of solutions corresponding to analytic data on the singularity which depend on the same number of free functions as the general solution. The next level is the corresponding statement with smooth rather than analytic data. The final one is to show that the solutions obtained in the second step include all those evolving from a non-empty open set of initial data on a regular Cauchy surface. Proofs on all these levels are available in certain examples.

One of the advantages of Fuchsian techniques is that they do not depend on

the spatial dimension. Thus it is possible to get results on hyperbolic equations in any space dimension which could not be achieved by the usual approaches to hyperbolic equations available today. In general relativity this means that results may be obtained on spacetimes with only one Killing vector or none at all. Techniques available up to now for studying the structure of spacetimes determined by Cauchy data belonging to a certain class near their singularities depended on having at least two Killing vectors so that the effective space dimension of the system of partial differential equations was one.

Potential applications of these techniques in general relativity to cases with little or no symmetry will now be listed. A case where this approach has been carried out successfully is that of isotropic singularities. The case of a perfect fluid with radiation equation of state has been analysed by Claudel[3] and generalizations to other fluids and to kinetic theory have been obtained by Tod[10] and Anguige[1]. Cases which have not yet been worked out are those of the Einstein-scalar field system in four dimensions (quiescent cosmology), the Einstein vacuum equations in sufficiently high dimensions, and vacuum solutions in four dimensions with polarized $U(1)$ -symmetry.

It may be asked why, if the Fuchsian approach is applicable without symmetry assumptions, the above examples are difficult to handle. The reason is that while the analytical theory is independent of symmetry, the algebra required to reduce a particular example to Fuchsian form may be very complicated.

To conclude, some examples will be mentioned where simple singularities have been successfully treated. The first is that of the Gowdy spacetimes. In [6] it was shown that there exist Gowdy spacetimes with singularities of a very particular (velocity-dominated) form which depend on the maximum number of free analytic functions. Recall that the Gowdy spacetimes are characterized by the fact that they are solutions of the vacuum Einstein equations, that they have two commuting spacelike Killing vectors, that they are spatially compact and that the so-called twist constants vanish. The spatial compactness is not relevant to the results of [6]. Later a similar result was obtained in the case where the condition on the twist constants is dropped while the situation considered is specialized in another direction (restriction to polarized models)[4]

The conclusion is that first steps have been taken towards a rigorous mathematical treatment of ‘simple’ singularities with a number of examples already having been worked out. Moreover, there is a potential for significant progress in this area in the near future. Finally, it should be noted that the results presented in this section and the last indicate the prospect of a fruitful marriage of numerical and analytical techniques in the study of the structure of singularities in inhomogeneous cosmological models.

References

- [1] Anguige, K.: Isotropic cosmological singularities. DPhil Thesis, University of Oxford, 1997.
- [2] Belinskii, V. A., Khalatnikov, I. M. and Lifschitz, E. M.: A general solution of the Einstein equations with a time singularity. *Adv. Phys.* 13, 639-667 (1982).
- [3] Claudel, C. M.: The Cauchy problem for quasi-linear hyperbolic evolution problems with a singularity in the time. Preprint, to appear in *Proc. R. Soc. Lond.*
- [4] Isenberg, J. and Kichenassamy, S.: Asymptotic behaviour in polarized T^2 -symmetric vacuum spacetimes. Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig, Preprint No. 13.
- [5] Kichenassamy, S.: *Nonlinear Wave Equations*. Marcel Dekker, New York, 1996.
- [6] Kichenassamy, S. and Rendall, A. D.: Analytic description of singularities in Gowdy spacetimes. Preprint, to appear in *Class. Quantum Grav.*
- [7] Leblanc, V. G., Kerr, D. and Wainwright, J.: Asymptotic states of magnetic Bianchi VI_0 cosmologies. *Class. Quantum Grav.* 12, 513-541 (1995).
- [8] Rendall, A. D.: Global dynamics of the mixmaster model. *Class. Quantum Grav.* 14, 2341-2356 (1997).
- [9] Rendall, A. D.: Solutions of the Einstein equations with matter. In M. Francaviglia, G. Longhi, L. Lusanna, E. Sorace (eds.) *Proceedings of the 14th International Conference on General Relativity and Gravitation*, pp.313-335. World Scientific, 1997.
- [10] Tod, K. P.: Isotropic cosmological singularities. (to appear in the *Proceedings of the XIIth International Congress of Mathematical Physics*, Brisbane, 1997.)
- [11] Wainwright, J. and Ellis, G. F. R. (eds.): *Dynamical systems in cosmology*. Cambridge University Press, Cambridge, 1997.
- [12] Weaver, M., Isenberg, J., and Berger, B. K.: Mixmaster behaviour in inhomogeneous cosmological models. *Phys. Rev. Lett.* 80, 2984-2987 (1998).